Further Properties of a Set of Symmetric Maxwellian Equations

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A recently proposed set of symmetric maxwellian equations with a quantum-mechanical coupling term between matter and fields is investigated further. Monopole and dipole solutions are described, from which the momentum of a moving charge is calculated. An alternative derivation of the Dirac quantization condition for monopoles is given.

Key words: Dirac Electron, Magnetic Monopole, Maxwell's Equations.

1. Introduction

In a recent article [1], we have proposed a set of symmetric maxwellian equations with source terms representing a quantum-mechanical coupling between matter and fields. The relativistic transformation properties and conservation laws of this set of equations were presented. Dual solutions were described as the electromagnetic fields of electric and magnetic monopoles, respectively. By way of example, the electric monopole was shown to have properties consistent with those of the Dirac electron.

In this article, we generalize the monopole solutions of [1] and, in addition, introduce a dipole-like solution. We then show (as already pointed out in [1]) that a linear combination of these two solutions yields the correct result $p = m v$ for the momentum of a moving charge calculated according to Eq. (5) of [1]. This is in contrast to the classical treatment, which yields (the wrong result) $p = 4/3\, m v$ [2].

We also examine briefly the monopole and dipole fields in the limit where the particle mass vanishes. Finally, the vector potentials of the monopole fields are given. The corresponding quantization condition is then derived and shown to be identical with the Dirac quantization condition for monopoles [3].

II. Generalized Monopole Fields

In [1], we proposed the following set of maxwellian equations:

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= -\boldsymbol{k}_0 \cdot \boldsymbol{B}, \qquad & \nabla \times \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{k}_0 \times \boldsymbol{B}, \\ \nabla \cdot \boldsymbol{B} &= -\boldsymbol{k}_0 \cdot \boldsymbol{E}, \qquad & \nabla \times \boldsymbol{B} &= \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{k}_0 \times \boldsymbol{E}, \end{aligned} \tag{1}$$

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where $\boldsymbol{k}_0$ is a radial vector at the point P where the fields are calculated, and of magnitude $k_0 = m_0 c/\hbar$, the inverse of the Compton wavelength for a particle of mass m_0. Dual, monopole-like solutions were then given ((7) and (8) of [1]), with energies $E \equiv \hbar\omega = \pm m_0 c^2$.

These solutions can be generalized for energies

$$E \equiv \hbar\omega = \pm\sqrt{(m_0 c^2)^2 + (\hbar k c)^2},$$

where $k = 2\pi/\lambda$ is the particle wave vector and λ the de Broglie wavelength, so that $p = \hbar k = m v$ is the particle momentum. The fields of the electric monopole are given in this case by

$$\begin{aligned} E_r &= (q/4\pi r^2)\cos kr \sin\omega t, \\ B_\theta &= (k_0 q/4\pi r)\, f^{\pm}(\theta)\cos kr \sin\omega t, \\ B_\varphi &= -(\omega q/4\pi c r)\, f^{\pm}(\theta)\cos kr \cos\omega t, \\ E_\theta &= -(k q/4\pi r)\, f^{\pm}(\theta)\sin kr \sin\omega t \end{aligned} \tag{2}$$

with $f^{\pm}(\theta) = (\cos\theta \pm 1)/\sin\theta$. They reduce to (7) of [1] in the limit $k \to 0$.

If we assume as in [1] that the fields do not extend beyond a spherical boundary of radius R_0, it then appears natural to require that the tangential field component E_θ vanish at $r = R_0$. This yields $k = n\pi/R_0$ $(n = 0, 1, 2, \ldots)$ which, for very large R_0, is a near continuum of k values.

Similar results are obtained by duality for the magnetic monopole.

As a final remark, we note that the generalized monopole fields described above are standing wave patterns, which could alternatively be written as superpositions of outgoing and incoming waves.

III. Dipole Fields

The set (1) admits also of another type of solution, which is the counterpart to the radiative dipole solu-

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tions of classical electromagnetic theory [4]. It can be verified that the following dipole fields are a solution of (1):

$$\tilde{B}_\varphi = (p_d k)\,[\cos k r/4\pi r^2 + k \sin k r/4\pi r]\sin\theta \sin\omega t,$$
$$\tilde{E}_r = -(2c/\omega r)\,\tilde{B}_\varphi \operatorname{cotg}\theta \operatorname{cotg}\omega t,$$
$$\tilde{E}_\theta = -(c/\omega)(p_d k)\,[\cos k r/4\pi r^3 + k \sin k r/4\pi r^2 - k^2 \cos k r/4\pi r]\sin\theta\cos\omega t,$$
$$\tilde{B}_\theta = (c k_0/\omega)\,\tilde{B}_\varphi \operatorname{cotg}\omega t,$$
$$\tilde{B}_r = 0; \quad \tilde{E}_\varphi = 0, \tag{3}$$

where $\hbar k c = p c = \pm\sqrt{(\hbar\omega)^2 - (m_0 c^2)^2}$. The quantity p_d is a dipole moment which can in principle take any value in view of the linearity of (1).

It is taken here as $p_d = q r_0$ (where $r_0 = \hbar/m_0 c$) to make the connection with classical electromagnetic theory, so that the azimuthal field component $\tilde{B}_\varphi$ is proportional to $(\hbar k/m_0)/c$.

Like in the monopole case of the previous section, the dipole fields are also standing wave patterns resulting from the superposition of outgoing and incoming waves.

It is also easy to verify that the above solution reduces to the classical dipole fields [4] in the limit $m_0 \to 0\,(k \to \omega/c)$.

IV. Momentum of a Uniformly Moving Charge

In [1], we had calculated the instantaneous momentum (the *Zitterbewegung*) of the electric monopole and found that it is purely electromagnetic in nature, i.e. only the first term in the expression for the local momentum density ((5) of [1])

$$\boldsymbol{p} = 2(\boldsymbol{E}\times\boldsymbol{B})/c - 2k_0 \int_0^t 2(\boldsymbol{E}\cdot\boldsymbol{B})\,\mathrm{d}t \tag{4}$$

makes a contribution. In this section, we extend this treatment to include the generalized monopole solution (2) and the dipole solution (3).

If we call $\boldsymbol{F}_M$ the monopole field (2) and $\tilde{\boldsymbol{F}}_d$ the dipole field (3), the field $\boldsymbol{F}_T = (\boldsymbol{F}_M - \gamma\tilde{\boldsymbol{F}}_d)$, where γ is a coupling constant, is also a solution because of the linearity of (1).

Using (4) for the local density of linear momentum, one can then proceed to calculate the time-averaged linear momentum component $\langle \boldsymbol{P}_\Omega \rangle$ of the moving charge along the Ω-axis (Fig. 1 of [1]), for the field $\boldsymbol{F}_T$ between hypothetical lower and upper radial bounds $r = r_p$ and $r = R_0$ [1]. After performing the time integral in (4) and the time averaging, and discarding terms which yield a null contribution owing to their time or spatial dependence, one then finds the following expression for $\langle \boldsymbol{P}_\Omega \rangle$:

$$\langle \boldsymbol{P}_\Omega \rangle = \frac{1}{c}\int \mathrm{d}\tau \left\{ \gamma(E_r \tilde{B}_\varphi + \tilde{E}_r B_\varphi)\sin\theta + \frac{c k_0}{\omega}\gamma(\tilde{E}_\theta B_\theta + E_\theta \tilde{B}_\theta)\cos\theta \right\}. \tag{5}$$

This is a bilinear product (of the space-dependent parts) of the monopole and dipole fields as given in (2) and (3), respectively. The quantity $\mathrm{d}\tau = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$ is the volume element.

A choice must now be made for the coupling constant γ introduced above which measures the contribution of the dipole field $\tilde{\boldsymbol{F}}_d$ to the total field $\boldsymbol{F}_T$ in the momentum calculation. We shall take $\gamma = (r_p/a)$, where a is the classical electron radius $a = \alpha r_0$ with α the electromagnetic fine structure constant. If r_p is the Planck length [1], γ is extremely small. It will nevertheless yield finite contributions since r_p is also the lower integration limit. This choice for γ ensures in addition that the integrals in (5) do not diverge even if $r_p \to 0$.

In the long wavelength limit, i.e. for de Broglie wavelengths which are large compared to both the particle's Compton wavelength r_0 and the lower bound r_p, the following is then found after integration. Each of the first two terms in (5), which originate from the vector product $(\boldsymbol{E}\times\boldsymbol{B})$ in expression (4) for the momentum density and represent the pure electromagnetic contribution, yields a momentum contribution of $2/3\,\hbar k$. Their sum, $4/3\,\hbar k$, is the result of classical electromagnetic theory [2]. The remaining terms in (5), however, which represent the contribution of the matter momentum (the last term in (4)), yield a compensating contribution, $-1/3\,\hbar k$. The final result is therefore the correct momentum, $\hbar k$. Corrections to the above derivations are on the order of $(k/k_0)^2$ and $(k r_p)$, or smaller.

V. Monopole and Dipole Fields in the Zero-Mass Limit

In the limit where the particle's mass $m_0 \to 0$, the generalized monopole fields in (2) are such that $E/c \equiv h\omega/c = \pm\hbar k$, and $B_\theta = 0$. At large distances (in the wave zone), this massless monopole behaves therefore like a transverse electromagnetic field with components (E_θ, B_φ), but still exhibiting the particular θ-dependence $f^{\pm}(\theta) = (\cos\theta \pm 1)/\sin\theta$. It can easily be verified that this field has thus still an angular momentum component $\mp h/2$, pointing in the direction opposite to its momentum component $\pm \hbar k$. It is therefore left-handed for positive energy.

Thus, the set of equations (1) admits two distinct types of solutions, namely the monopole (2), and the dipole (3). As already pointed out in Section III, the dipole fields smoothly reduce to their classical counterpart in the zero-mass limit. In contrast, the monopole reduces in the zero-mass limit to a quantized, spin-1/2, neutrino-like field.

VI. Vector Potential of the Magnetic Monopole. The Dirac Quantization Condition.

It can be verified that the fields of the magnetic monopole ((8) of [1]) can be derived from a magnetic vector potential A such that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}. \tag{6}$$

The corresponding components of A are thus given by

$$\begin{aligned} A_r &= 0, \\ A_\theta &= (c\,k_0/\omega)\,(g/4\pi r)\,f^{\pm}(\theta)\cos\omega t, \\ A_\varphi &= -(g/4\pi r)\,f^{\pm}(\theta)\sin\omega t, \end{aligned} \tag{7}$$

where $f^{\pm}(\theta) = (\cos\theta \pm 1)/\sin\theta$ and $\omega = \pm\omega_0 = \pm m_0 c^2/\hbar$.

A similar result is obtained by duality for the electric monopole.

One first observes that the components B_r and A_φ are the same as those familiar from the standard treatment of the quantization condition [3]. One could then proceed from there and apply the usual requirement for single valuedness of the wavefunction under a gauge transformation [3], and thus obtain the quantization condition $q\,g/4\pi\hbar c = \pm n/2$, where n is an integer.

An alternative derivation can, however, be made by considering the action of the azimuthal electric field component E_φ of the magnetic monopole on an electric monopole charge. First, using the electric field E_r of an electric monopole from (7) of [1], one finds that such a charge is pulsating and given by $Q = q\sin\omega_0 t$. The magnetic charge g itself is located at the center of the coordinate system (Fig. 1 of [1]) whereas the electric charge is placed at a large distance $r \gg r_0 (= \hbar/m_0 c)$. Under the influence of the electric field E_φ of the magnetic monopole, the electric charge then rotates on an azimuthal loop centered on the Ω-axis (Fig. 1 of [1]) and acquires thus a momentum P_φ obtained from (6) and (7), namely

$$P_\varphi = (Q/c)\,A_\varphi = -(q\,g/4\pi\hbar c)\,\frac{\cos\theta \pm 1}{r\sin\theta}\,\hbar\sin^2\omega_0 t\,. \tag{8}$$

The limit of this expression as the loop radius $r\sin\theta \to r_0/2$ (the amplitude of the *Zitterbewegung*) and $r \gg r_0$ is then zero or $\pm 4(q\,g/4\pi\hbar c)\,\hbar k_0 \sin^2\omega_0 t$, where $k_0 = 1/r_0$. The time-averaged electron momentum on the azimuthal loop is then zero or

$$\langle P_\varphi \rangle = \pm 2(q\,g/4\pi\hbar c)\,\hbar k_0\,. \tag{9}$$

On the other hand, it was found in [1] that the spin of the electron (which is a result of its *Zitterbewegung*) is given by

$$\langle M_\Omega \rangle = -(r_0/2)\langle P_\varphi \rangle = \mp \hbar/2\,, \tag{10}$$

so that

$$\langle P_\varphi \rangle = \pm \hbar k_0\,. \tag{11}$$

Comparison of (9) with (11) yields at once $q\,g/4\pi\hbar c = 1/2$.

The monopole quantization condition follows thus from the requirement that the electron spin remain unchanged in the presence of the magnetic monopole. This criterion is basically equivalent to the above cited requirement of univaluedness of the wavefunction, but formulated here in the language of electromagnetic theory.

VII. Conclusion

In [1] and in this article, we have proposed and investigated the properties of a new set of symmetric maxwellian equations. Dual electric and magnetic monopole solutions, as well as dipole solutions have been described. By imposing a constraint of regularity on the monopole fields, a boundary condition $\alpha \ln(R_0/r_p) \equiv 1$ has been obtained, which has suggested a lower radial bound r_p on the order of the Planck length. The correct expression for the linear momentum of a moving charge, $p = \hbar k$, has been obtained by appealing to a dipole field with a coupling constant $\gamma = r_p/a$, suggesting the necessity of a small but nevertheless finite coupling between electricity and gravitation to properly account for the particle momentum. Finally, an alternative derivation of the Dirac quantization condition has been given.

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